MATH 2710: NOTES FOR ANALYSIS

The main ideas we will learn from analysis center around the idea of a limit. Limits occurs in several settings. We will start with finite limits of sequences, then cover infinite limits, and then look at limits of functions. We have not given a proper definition of $\mathbb{R}$, so we can't really treat limits completely because they may not exist just in $\mathbb{Q}$. However, all the relevant definitions make sense in $\mathbb{Q}$. I will note in the text where the situation is different between $\mathbb{Q}$ and $\mathbb{R}$.

Sequences and limits of sequences

A sequence is just a function with domain $\mathbb{N}$. We will mostly be interested in sequences with values in the real numbers. The usual function notation $a : \mathbb{N} \to \mathbb{R}$, which would list the values in the sequence as $a(1), a(2), a(3), \ldots$ is often not the most convenient way to write a sequence. Instead we write the argument as a subscript, so the foregoing would read: $a_1, a_2, a_3, \cdots$. Sometimes we write the sequence as $\{a_n\}$ or $\{a_n\}_1^\infty$.

Like functions, sequences are often written as rules.

(1) The rule $a_n = n^2$ defines a sequence with values $1, 4, 9, 16, 25, \ldots$
(2) We could write $\{\frac{1}{n}\}$ for the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
(3) Often, a sequence is given by writing a few terms and leaving the reader to infer a rule, e.g. writing $2, 3, 5, 7, 11, 13, \ldots$ to indicate the sequence of primes. This is imprecise, but we will still do it from time to time.

The prototype of a convergent sequence is something like $\{\frac{1}{n}\}$, where it seems that the values get arbitrarily close to zero and stay close to zero as n increases. The notion of *closer* is measured using the distance (in the real numbers).

Definition 1. The distance between $x, y \in \mathbb{R}$ is $|x - y|$, where the absolute value is defined to be

$$|w| = \begin{cases} w & \text{if } w \geq 0, \\ -w & \text{if } w < 0. \end{cases}$$

Now we have a distance we can talk about x being closer to y than z is to y, meaning $|x - y| < |z - y|$. If $\epsilon > 0$ we can also talk about x being closer than ϵ to y, meaning $|x - y| < \epsilon$. We also make a note of the most useful property of distance; we record it as a lemma.

Lemma 2 (Triangle Inequality)**.** *For points $x, y, z \in \mathbb{R}$ we have $|x - y| \leq |x - z| + |z - y|$.*

Intuitively this says the distance from x to y is no larger than the distance from x to y via z.

Proof. This is a straightforward computation using cases. Clearly the points are arranged on a line. Swapping x and y does not change the expression, so we can assume $x > y$ (the case $x = y$ is obvious). The cases are then $z \geq x$, $x > z > y$, $y \geq z$. In all cases $|x - y| = x - y$. What changes is the right side. If $z \geq x$ then $|z - y| \geq |x - y|$, so the inequality is true. If $y \geq z$ then also $|x - z| \geq |x - y|$ so the inequality is true. Finally, if $x > z > y$ then $|x - z| = x - z$ and $|z - y| = z - y$ so $|x - z| + |y - z| = x - z = |x - z|$. This completes the proof. $\square$

A very useful version of the triangle inequality is as follows.

Lemma 3. *For points $a, b \in \mathbb{R}$ we have $||a| - |b|| \leq |a - b|$.*

Proof. The triangle inequality says $|x+y| \leq |x|+|y|$, so $|x+y|-|y| \leq |x|$. Now put $x+y=a$ and $y=b$, so $x=a-b$. We obtain $|a|-|b| \leq |a-b|$. Repeating this for $x+y=b$, $y=a$ so $x=b-a$ we have $|b|-|a| \leq |b-a| = |a-b|$. But $||a|-|b||$ is equal to one of $|a|-|b|$ or $|b|-|a|$, both of which we showed are no larger than $|a-b|$, completing the proof. □

The idea of a sequence x_n converging to a number L is supposed to capture that x_n gets arbitrarily close to L as n gets large. The notion of arbitrarily close clearly requires a quantifier: there is no specific proximity we are looking for, but instead we are looking to get the x_n closer to L than any postive number ϵ. The notion of n being large also requires a quantifier. However, a little thought shows us that for a specfic tolerance $\epsilon > 0$ the distance we have to go out along the sequence to get $|x_n - L| < \epsilon$ depends both on ϵ and on the sequence. For example, with $x_n = \frac{1}{n}$, $L = 0$ and $\epsilon = 0.1$ we need $n > 10$ to get $|x_n - L| < \epsilon$, while for $\epsilon = 0.01$ we would need $n > 100$; if the sequence was instead $y_n = n^{-2}$ then to get $|y_n - L| < 0.1$ we'd only need $n > 3$ and to get $|y_n - L| < 0.01$ we would need $n > 10$. The fact that the quantifier for how far out in the sequence we go can depend on ϵ means it should be determined after ϵ is determined.

The above shows that for every $\epsilon > 0$ we want that there is a notion of *far enough out* in the sequence, namely a N, possibly depending on ϵ, such that going any further out, meaning taking $n > N$, gives $|x_n - L| < \epsilon$. This is what we use to make the formal definition of a limit. Before we give this definition, one other important point: if you don't understand the reasoning for defining the limit in this way, it is not essential to do so. Some people use the definition for a long time before they really "get it". If it doesn't make sense to you intuitively, don't worry about it – just learn the definition and practice using it

Definition 4. The sequence a_n converges to L if $\forall \epsilon > 0\ \exists N$ such that $n \geq N \implies |x_n - L| < \epsilon$. This is often written as $\lim_{n\to\infty} a_n = L$ or as $a_n \to L$ when $n \to \infty$.

From this we can make two other definitions. The first is that a sequence converges if there is some number to which it converges, while the second is that otherwise it diverges.

Definition 5. A sequence a_n is convergent if there is an L such that $a_n \to L$ when $n \to \infty$. If a_n is not convergent it is called divergent.

Remark 6. The issue of whether a sequence converges is where $\mathbb{Q}$ and $\mathbb{R}$ are very different. Even if all the numbers we use in our definitions, except L, are from $\mathbb{Q}$, there are sequences that converge in $\mathbb{R}$ and do not converge in $\mathbb{Q}$. A simple example: in a homework problem you saw how given a positive number a you could construct a number $\tilde{a}$ so that $\tilde{a}^2$ is closer to 2 than a^2. This can be used recursively to construct a sequence x_j in $\mathbb{Q}$ such that $x_j^2 \to 2$. Of course x_j does not converge in $\mathbb{Q}$, because the limit L would have to have $L^2 = 2$ and we also proved there is no such $L \in \mathbb{Q}$. But it turns out one can prove that $x_j \to \sqrt{2}$ in $\mathbb{R}$.

Now we are going to do some simple proofs that specific sequences converge to specified values. In each case the proof begins by saying that $\epsilon > 0$ and then says how to choose N (depending on ϵ) so as to ensure that $n > N \implies |x_n - L| < \epsilon$. However, the point is to learn how to do these yourself, so each proof is preceeded by a description of the thought process used to make the choice I gave. Note that the choice of N in each case is not unique – any larger N would have worked – so the reasoning I give is not unique either. Your reasoning for a problem might be different to mine, which is fine so long as it works.

Example 7. Prove $\{\frac{1}{n}\}$ converges to 0 as $n \to \infty$.
My reasoning: Given $\epsilon > 0$ I want N so $n > N$ will make $|x_n - 0| = \frac{1}{n} < \epsilon$. To get this I need $n > \frac{1}{\epsilon}$, so I take $N > \frac{1}{\epsilon}$.
Proof: Given $\epsilon > 0$ let $N > \frac{1}{\epsilon}$. If $n > N$ then $|x_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$, so $\frac{1}{n} \to 0$.

Example 8. Prove $\{\frac{4n}{n+3}\}$ converges to 4 as $n \to \infty$.
My reasoning: With $x_n = \frac{4n}{n+3}$ we have $|x_n - 4| = \left|\frac{4n-4(n+3)}{n+3}\right| = \frac{12}{n+3}$ which will be less than ϵ if $n+3 > \frac{12}{\epsilon}$. It is tempting to take $N > \frac{12}{\epsilon} - 3$, so $n > N$ implies $n+1 > \frac{12}{\epsilon}$ and $\frac{12}{n+3} < \epsilon$, but it is just as simple and perhaps a little neater to take $N > \frac{12}{\epsilon}$ and say $n > N$ implies $\frac{12}{n+3} < \frac{12}{N} < \epsilon$.
Proof: Given $\epsilon > 0$ take $N > \frac{12}{\epsilon}$. Then $n > N$ implies $|x_n - 4| = \left|\frac{4n-4(n+3)}{n+3}\right| = \frac{12}{n+3} < \frac{12}{N} < \epsilon$. So $x_n \to 4$.

Example 9. Prove $\frac{3}{n^3}$ converges to 0 as $n \to \infty$.
Reasoning: We need N so $n > N$ implies $\frac{1}{n^3} < \epsilon$. Evidently $N > \left(\frac{3}{\epsilon}\right)^{1/3}$ will work, so we can use that. An alternative would be to say that if $n^2 > 3$, for which we only need $n > N > 2$ we would have $n^3 > 3N$ so $\frac{3}{n^3} < \frac{1}{N} < \epsilon$ if $N > \frac{1}{\epsilon}$. I prefer this latter way, which requires $N > \max\{2, \frac{1}{\epsilon}\}$ only because the method is more flexible.
Proof: Given $\epsilon > 0$ take $N > \max\{2, \frac{1}{\epsilon}\}$. Then $N^2 > 4$ so $n^3 > 3n$ and $\frac{3}{n^3} < \frac{1}{n} < \epsilon$ which shows the result.

Example 10. Prove $\{\frac{n+1}{n^2-3}\}$ converges to 0.
Reasoning: We need $\frac{|n+1|}{|n^2-3|} < \epsilon$. It it a bit messy to figure out what the best choice of N so $n > N$ would make this happen – we'd have to solve a quadratic equation – but we don't need the best N we just need one that works. Accordingly we make some crude inequalities that bound the numerator and denominator just like we did in the previous problem. Notice $|n+1| < 2n$ for $n > 1$. Also, lets look for where $|n^2 - 3| > \frac{n^2}{2}$. This occurs when $n^2 > 6$, so when $n > 2$. If we have those two inequalities and also $\frac{2n}{(n^2/2)} < \epsilon$, meaning $n > \frac{4}{\epsilon}$, then we get the inequality we want. So we take $N > \max\{2, \frac{4}{\epsilon}\}$.
Proof: Given $\epsilon > 0$ let $N > \max\{2, \frac{4}{\epsilon}\}$. If $n > N$ then $n > 2$ which implies $|n+1| < 2n$ and $n^2 > 6$, hence $n^2 - 3 > \frac{n^2}{2}$. Then $\frac{|n+1|}{|n^2-3|} < \frac{4n}{n^2} = \frac{4}{n} < \epsilon$, showing the result.

At this point it might be worth summarizing the structure that appeared in all these proofs. When giving a proof of a limit of a sequence from the definition, you can always use the following structure:
Given $\epsilon > 0$ here is a rule for getting N (usually as a function of ϵ) and an argument showing that this choice of N ensures that $|a_n - L| < \epsilon$.

It is now about time to prove that some sequences are divergent. The basic idea is to use contradiction: assume that the sequence is convergent and derive a contradiction. This works well on sequences that diverge to infinity, but also on sequences that oscillate around without settling down in the real numbers. One way to have the latter occur is if the sequence appears to have parts that are converging to different values. We will see examples of both types. In both cases the key is that if the sequence did converge then all x_n values are eventually close together, so if this is not true the sequence must diverge.

Example 11. Prove that $\{(-1)^n\}$ does not converge as $n \to \infty$.
Reasoning: No matter how far you go out in the sequence $-1, 1, -1, 1, -1, \dots$ you have points 1 and -1 that are separated by distance 2. If x_n is within ϵ of L for all n then L is within ϵ of both 1 and -1. But the triangle inequality allows us to say that the distance between -1 and 1 is at most the distance via L, which is $|-1-L| + |1-L|$ so $2 < 2\epsilon$ which gives $\epsilon > 1$. If we take $\epsilon = 1$ we should get a contradiction.
Proof: If the sequence converged to L then for $\epsilon = 1$ there is N so $n > N$ implies $|x_n - L| < 1$. But then $2 = 2\epsilon > |x_n - L| + |x_{n+1} - L| = |-1-L| + |1-L| \geq |-1-1| = 2$ by the triangle inequality, a contradiction.

In the previous example it looks like this sequence has two pieces: the odd terms are all -1 so are converging to -1 and the even terms are all 1 so are converging to 1. The issue is that the sequence can't converge to two different places. The idea of the sequence having pieces that do these two different things can be made precise using the idea of subsequences and limit points. We won't do that right now, but we will repeat the above argument to show that, in fact, it is always impossible for a sequence to converge to two places.

Lemma 12. *A convergent sequence has a unique limit.*

Reasoning: we use a small modifiation of the argument used in the previous exercise. The idea is that the sequence can't simultaneously converge to two different points L and L' because these points are a positive distance apart, so it is not possible to be simultaneously arbitrarily close to both of the points. The issue is to make this clear with some sort of ϵ and N. It may help to draw a number line showing L and L'. When you do so you will see that it is not possible to be within $\frac{1}{2}|L-L'|$ of both L and L', so this is what we take for ϵ in order to get a contradiction.

Proof. Suppose $a_n \to L$ and also $a_n \to L'$ as $n \to \infty$. If $L \neq L'$ take $\epsilon = \frac{1}{2}|L-L'| > 0$ and get N_1 so $n > N_1$ implies $|x_n - L| < \epsilon$ and N_2 so $n > N_2$ implies $|x - L'| < \epsilon$. Then for $n > N = \max\{N_1, N_2\}$ we have by the triangle inequality

$$|L - L'| \leq |L - x_n| + |L' - x_n| < \epsilon + \epsilon = |L - L'|$$

which is a contradiction. We conclude $L = L'$, so the limit is unique. □

Example 13. Prove that $\{n^2 - 1\}$ does not converge as $n \to \infty$.
Reasoning: If it converged to L then for any tolerance ϵ it would eventually be within ϵ of L. If this happened then $n > N$ would mean $n^2 - 1$ and $(n+1)^2 - 1$ would be within 2ϵ of each other, but the difference between them is $2n + 2 > 2$. So the condition for convergence will fail if $\epsilon < 1$.
Proof: If the sequence converged to L then for $\epsilon = \frac{1}{2}$ we could take N so $n > N$ would imply both $|n^2 - 1 - L| < \epsilon$ and $|(n+1)^2 - 1 - L| < \epsilon$. But then using the triangle inequality $|n^2 - 1 - (n+1)^2 + 1| < 2\epsilon = 1$, so $|2n + 1| < 1$ when $n > N$, which is a contradiction.

For the previous example, if you were given the problem in a calculus class you would have been expected to say that the limit is ∞. Obviously we can't prove this using the definition we have for sequences with finite limits, because the notion of $|x_n - \infty| < \epsilon$ does not make sense – the symbol ∞ does not refer to a number, and the distance we are using is not defined if one of the quantities is infinite. Instead, what $x_n \to \infty$ is supposed to mean is that the values get arbitrarily large, arbitrarily meaning larger than any fixed M. The expression $x_n \to -\infty$ has a similar meaning.

Definition 14. We say $x_n \to \infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = \infty$ if $\forall M\ \exists N$ such that $n > N \implies x_n > M$. We say $x_n \to -\infty$ or $\lim_{n\to\infty} x_n = -\infty$ if $\forall M\ \exists N$ such that $n > N \implies x_n < M$.

Example 15. Prove that $\{n^2 - 1\}$ converges to ∞ as $n \to \infty$.
Reasoning: Given M we need to show that $n^2 - 1$ is bigger than M. To do so we need to say $n^2 - 1$ is larger than something. One way is to say $n^2 - 1 \geq 2n - 1 > n$ if $n > 2$, and thus $n^2 - 1 > M$ if $N > M$. So take $N = \max\{2, M\}$ to get the right inequalities.
Proof: Given M let $N = \max\{2, M\}$. If $n > N$ then $n > 2$ so $n^2 - 1 \geq 2n - 1 > n > N \geq M$.

Algebraic properties of limits

In calculus you would have been taught certain basic rules about limits that were used to compute limits of complicated limits from simpler limits. They are established in the following theorems, which are collectively referred to as the limit laws.

Theorem 16. *If $a_n \to a$ and $b_n \to b$ then $(a_n + b_n) \to (a + b)$.*

Reasoning: We know $|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n + b|$. If we want this latter to be less than a given $\epsilon > 0$ we could make both $|a_n - a| < \frac{\epsilon}{2}$ by making $n > N_1$ and $|b_n - b| < \frac{\epsilon}{2}$ by making $n > N_2$. Then $n > \max\{N_1, N_2\}$ would make both true and give us the result.

Proof. Given $\epsilon > 0$ use the assumed limits to take N_1 so $n > N_1$ implies $|a_n - a| < \frac{\epsilon}{2}$ and N_2 so $n > N_2$ so $|b_n - b| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ so $n > N$ implies both $n > N_1$ and $n > N_2$, whence $|a_n - a| < \frac{\epsilon}{2}$ and $|b_n - b| < \frac{\epsilon}{2}$ so

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Theorem 17. *If $a_n \to a$ and $b_n \to b$ then $a_n b_n \to ab$.*

Reasoning: The key to this computation is that

$$|a_n b_n - ab| = |a_n(b_n - b) + b(a_n - a)| \leq |a_n||b_n - b| + |b||a_n - a|.$$

We can take N_1 so $n > N_1$ implies $|a_n - a| < \frac{\epsilon}{2|b|}$ and therefore the second term less than $\frac{\epsilon}{2}$. We'd like to deal with the first term in a similar way but in that case both factors depend on n, so we'd want to $|a_n|$ to be bounded by something. However, since $a_n \to a$ we can see that $|a_n| \leq |a| + 1$ if n is large enough. Specifically, if $\epsilon = 1$ then there is N_2 so if $n > N_2$ then $|a_n - a| < 1$ and thus $|a| - 1 \leq |a_n| \leq |a| + 1$. Moreover, there is N_3 so if $n > N_3$ then $|b_n - b| < \frac{\epsilon}{2(|a|+1)}$. If $n > N = \max\{N_1, N_2, N_3\}$ then we should get the result. Along the way it seems we proved the following result which will be useful later as well.

Lemma 18. *If $a_n \to a$ then there is N so $n > N$ implies $|a| - 1 \leq |a_n| \leq |a| + 1$.*

Proof. Take $\epsilon = 1$ and N so $n > N$ implies $|a_n - a| < 1$. Then $|a| - 1 < |a_n| < |a| + 1$. □

Proof of theorem. Given $\epsilon > 0$ use the assumed limit of a_n to take N_1 so $n > N_1$ implies $|a_n - a| < \frac{\epsilon}{2|b|}$. Using the lemma and $a_n \to a$ take N_2 so $n > N_2$ so $|a_n| < |a| + 1$. Finally use the limit $b_n \to b$ to take N_3 so $n > N_3$ implies $|b_n - b| < \frac{\epsilon}{2(|a|+1)}$. Then

$$|a_n b_n - ab| = |a_n(b_n - b) + b(a_n - a)| \leq |a_n||b_n - b| + |b||a_n - a| \leq (|a| + 1)\frac{\epsilon}{2(|a| + 1)} + |b|\frac{\epsilon}{2|b|} = \epsilon.$$

□

A special case of the preceding is that if $b_n \to b$ then $-b_n \to -b$, because it is easy to prove that the constant sequence -1 converges to -1. Then the fact that one can add convergent sequences term by term and add the limits means one can do the same with differences. Specifically, $a_n \to a$ and $b_n \to b$ implies, as we just said, $-b_n \to -b$, and using the result about sums, $a_n - b_n \to a - b$. To do the result corresponding to division we have to work a little harder.

Theorem 19. *If $a_n \to a$ and $a \neq 0$ then $\frac{1}{a_n} \to \frac{1}{a}$.*

Reasoning: When we compute the difference between the sequence elements and the putative limit we find

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{1}{|a||a_n|}|a - a_n|$$

which looks promising because $|a - a_n|$ can be made small by making n large. However, to get this less than $\epsilon > 0$ we need to know a lower bound for $|a||a_n|$. There is no problem with $|a|$ because it is a non-zero constant. For $|a_n|$ we could use the lemma already established and find N_1 so $n > N_1$ implies $|a_n| \geq |a| - 1$, but this would not be useful unless we knew $|a| - 1 > 0$. We'd need something better, like $|a_n| > \frac{|a|}{2}$. But we can get this: just take $\epsilon = \frac{|a|}{2} > 0$ and then N_1 so $n > N_1$ implies $|a_n - a| < \epsilon = \frac{|a|}{2}$, from which $|a_n| > \frac{|a|}{2}$. Then $|a||a_n| > \frac{|a|^2}{2}$. We could then take N_2 so $n > N_2$ implies $|a_n - a| < \frac{\epsilon|a|^2}{2}$.

Proof. First take $\epsilon = \frac{|a|}{2}$ and use $a_n \to a$ to get N_1 so $n > N_1$ implies $|a_n - a| < \frac{|a|}{2}$ so that $|a_n| > \frac{|a|}{2}$. Then take N_2 so $n > N_2$ implies $|a_n - a| < \frac{\epsilon |a|^2}{2}$. For $n > N = \max\{N_1, N_2\}$ compute

$$\Big|\frac{1}{a_n} - \frac{1}{a}\Big| = \frac{1}{|a||a_n|}|a - a_n| < \frac{2}{|a|^2}\frac{\epsilon |a|^2}{2} = \epsilon$$

which proves the result. □

Obviously we can combine this with the previous theorem to obtain that $a_n \to a \neq 0$ and $b_n \to b$ implies $\frac{b_n}{a_n} \to \frac{b}{a}$.

At this point we can use these theorems and the fact that $\frac{1}{n} \to 0$ to obtain the results of the other exercises we have done so far which established convergence, as follows.

- We have $\frac{3}{n^3} = 3\frac{1}{n}\frac{1}{n}\frac{1}{n}$ is a product of a constant sequence and three sequences that converge to zero. By our product result applied three times we find $\frac{3}{n^3} \to 0$.
- Rewrite $\frac{n+1}{n^2-3} = \frac{(1/n)+(1/n)(1/n)}{1-3(1/n)(1/n)}$. Then the numerator converges to zero (by applying the product result and the sum result) and the denominator converges to 1 by the same argument. Using the ratio result we find that the sequence converges to 0.

In fact it is not hard to see that the limit of any rational function of n can be computed in this manner as $n \to \infty$, if the limit exists.

This does not mean that all limits can be done in this way. Here is an important one that cannot be done in this manner.

Lemma 20. *If $|r| < 1$ then $r^n \to 0$ as $n \to \infty$.*

Reasoning: Given $\epsilon > 0$ we need N so $n > N$ implies $|r|^n < \epsilon$. We don't know how to do this for general r because we don't know anything about the powers of r, but there is one situation in which we do know about powers – the situation in which we can apply the binomial theorem. Some thought suggests that this will let us prove the result in the special case where $r = \frac{p}{p+1}$, because then $(p+1)^n = \sum_0^n \binom{n}{j} p^{n-j} \geq np^{n-1}$. So

$$\big(\frac{p}{p+1}\big)^n \leq \frac{p^n}{np^{n-1}} = \frac{p}{n} \to 0 \text{ as } n \to \infty.$$

What is more, if $|r| < \frac{p}{p+1}$ this still proves $|r|^n \to 0$ as $n \to \infty$, and since the distance from $|r|$ to 1 is positive (because $|r| < 1$) and $\frac{p}{p+1} \to 1$ as $p \to \infty$ we can find a p so $\frac{p}{p+1}$ is in the interval $(|r|, 1)$, which makes $|r| < 1$. This is the idea of the proof.

Proof. We begin with the fact that $\frac{p}{p+1} \to 1$ as $p \to \infty$, which is easily established using the limit laws. Use this to find a p for which $\big|1 - \frac{p}{p+1}\big| < 1 - |r|$. Since $\frac{p}{p+1} < 1$ we conclude that $|r| < \frac{p}{p+1} < 1$.

Now $|r^n - 0| = |r|^n < \big(\frac{p}{p+1}\big)^n$. Given $\epsilon > 0$ take $N > \frac{p}{\epsilon}$ (notice that this is ok because we already decided on the value for p). Then compute $(p+1)^n \geq np^{n-1}$ using the binomial theorem and complete the proof by deducing

$$\big(\frac{p}{p+1}\big)^n \leq \frac{p}{n} \leq \frac{p}{N} < \epsilon.$$ □

One can also determine that $|r| > 1$ implies $|r|^n \to \infty$ as $n \to \infty$ by a similar method. Try it for yourself!

Series

We know how to add finitely many terms. Suppose we have a sequence $a_1, a_2, a_3, \dots$. We can make a new sequence as follows:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= \sum_{j=1}^{n} a_j \end{aligned}$$

If the sequence $s_1, s_2, s_3, \dots$ converged to a limit L, it would be natural to think of L as the result of the infinite sum $a_1 + a_2 + a_3 + \cdots$. We define this to be the case.

Definition 21. If the sequence $\{\sum_{j=1}^{n} a_j\}_{n=1}^{\infty}$ converges to L as $n \to \infty$ then we set $\sum_1^\infty a_j = L$ and call this a convergent series.

There are some basic examples of convergent series, as follows.

Example 22. Prove $\sum_1^\infty \frac{1}{j(j+1)} = 1$.
Reasoning: We consider $\sum_1^n \frac{1}{j(j+1)}$. Several approaches are possible. One is to recognize that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Then

$$\sum_1^n \frac{1}{j(j+1)} = \bigl(1 - \frac{1}{2}\bigr) + \bigl(\frac{1}{2} - \frac{1}{3}\bigr) + \bigl(\frac{1}{3} - \frac{1}{4}\bigr) + \bigl(\frac{1}{4} - \frac{1}{5}\bigr) + \dots.$$

Internal cancellation then suggests that the sum is $1 - \frac{1}{n+1}$. We should be able to prove it by induction. When $n = 1$ we get $\frac{1}{2}$. If the result was true for n then adding the next term we would get

$$1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1}\bigl(1 - \frac{1}{n+2}\bigr) = 1 - \frac{1}{n+2},$$

which verifies the inductive step. Using this and the fact that $\frac{1}{n+1} \to 0$ as $n \to \infty$ gives the result.

Proof. We first prove by induction that $\sum_1^n \frac{1}{j(j+1)} = 1 - \frac{1}{n+1}$. This is true for $n = 1$ where both sides are $\frac{1}{2}$. If it is true for n then

$$\sum_1^{n+1} \frac{1}{j(j+1)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+2},$$

so the formula is true for all $n \geq 1$ by induction.

Then use that $\frac{1}{n+1} \to 0$ as $n \to \infty$, which may be proved directly (given $\epsilon > 0$ take $N > \frac{1}{\epsilon}$ and $n + 1 > N > \frac{1}{\epsilon}$ implies the result) or by writing it as $\frac{(1/n)}{1+(1/n)}$ and using the limit laws. This implies $\sum_1^n \frac{1}{j(j+1)} \to 1$ as $n \to \infty$, establishing the convergence of the series and that its sum is 1. $\square$

Example 23. If $|r| < 1$ prove that $\sum_1^\infty r^n = \frac{1}{1-r}$.
Reasoning: This is done using a trick. Compute

$$(1 - r)\sum_1^n r^j = \sum_1^n r^j - \sum_1^n r^{j+1} = \sum_1^n r^j - \sum_2^{n+1} r^j = 1 - r^{n+1}.$$

Thus $\sum_1^n r^j = \frac{1-r^{n+1}}{1-r}$. This and the fact that $r^n \to 0$ as $n \to \infty$ proves the result.

Proof. Compute

$$(1-r)\sum_1^n r^j = \sum_1^n r^j - \sum_1^n r^{j+1} = \sum_1^n r^j - \sum_2^{n+1} r^j = 1 - r^{n+1}$$

and deduce $\sum_1^n r^j = \frac{1-r^{n+1}}{1-r}$. Use Lemma 20 from the end of the last section and the given fact $|r|<1$ to get $r^n \to 0$ as $n\to\infty$ and multiply by the constant $\frac{r}{1-r}$ to get $\frac{r^{n+1}}{1-r}\to 0$ as well. This tells us that $\lim_{n\to\infty}\sum_1^n r^j = \frac{1}{1-r}$, which is what we wanted to prove. □

There are some other basic facts one can prove about convergence of series. One of the most important is an easy test for divergence.

Theorem 24. *If $\sum_1^\infty r_j$ converges then $r_j \to 0$.*

Reasoning: The point here is that convergence of the series means the sequence $\sum_1^n r_j$ converges, so the terms of this sequence of partial sums are all close to a limiting value when n is large, which in turn means that all the terms must be close to one another. However the difference between consecutive terms for n and $n+1$ is precisely r_{n+1}, so r_{n+1} must also be small, in fact (by the triangle inequality) less than twice the distance from the partial sum to its limit value. What remains is to quantify this with ϵ and N.

Proof. The series converges to a limit L, so given $\epsilon>0$ there is N so $n>N$ implies $|\sum_1^n r_j - L| < \frac{\epsilon}{2}$. In particular, using $n>N$ and thus $n+1>N$, and the triangle inequality:

$$|r_{n+1}| = \Big|\big(\sum_1^{n+1} r_j - L\big) - \big(\sum_1^n r_j - L\big)\Big| \le \big|\sum_1^{n+1} r_j - L\big| + \big|\sum_1^n r_j - L\big| < \epsilon$$

so for $n>N$ we have $|r_{n+1}|<\epsilon$ which proves the result. □

It is a very important fact that you learned in calculus class that the converse of this result is false: there are sequences for which the terms go to zero but the series diverges. A basic example is the *harmonic series* $\sum_1^\infty \frac{1}{j}$, for several reasons, one of which is that the method of proof is useful elsewhere.

Theorem 25. *The series $\sum_1^\infty \frac{1}{j}$ diverges to ∞.*

Reasoning: The proof contains an important idea, which is that one can break the sum up into blocks of different lengths such that each block contributes at least a constant amount to the sum. The decomposition into blocks is as follows:

$$\begin{aligned}\sum_1^\infty \frac{1}{j} &= 1 + \frac{1}{2} + \big(\frac{1}{3}+\frac{1}{4}\big) + \big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\big) + \cdots \\ &\ge 1 + \frac{1}{2} + \big(\frac{1}{4}+\frac{1}{4}\big) + \big(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\big) + \cdots \\ &\ge 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots\end{aligned}$$

What we see is that adding the terms in the block from 2^{n-1} to 2^n we have 2^{n-1} terms each of size at least 2^{-n}, so the sum of this block contributes at least $\frac{1}{2}$ to the series. We can make this precise using induction, because this will show $\sum_1^{2^n} \ge 1 + \frac{n}{2}$. From here it is easy to show the series diverges as stated.

Proof. We prove by induction that $\sum_1^{2^n} \frac{1}{j} \geq 1 + \frac{n}{2}$ for $n \geq 1$. This is true for $n = 1$, and the inductive step is

$$\sum_{1}^{2^{n+1}} \frac{1}{j} = \sum_{1}^{2^n} \frac{1}{j} + \sum_{2^{n+1}-1}^{2^{n+1}} \frac{1}{j} \geq 1 + \frac{n}{2} + \sum_{2^{n+1}-1}^{2^{n+1}} 2^{-(n+1)} \geq 1 + \frac{n}{2} + 2^n 2^{-(n+1)} = 1 + \frac{n+1}{2}$$

so it is true for all $n \geq 1$.

Now given M take $N > 2^{2M}$. Then $n > N$ implies $\sum_1^N \frac{1}{j} \geq \sum_1^{2^{2M}} \geq 1 + M > M$, so the partial sums diverge to ∞. $\square$

Limits of functions

The limits you most frequently encounter in calculus class are limits of functions not sequences. In these notes we will limit ourselves to considering limits of the form $\lim_{x \to c} f(x) = L$ where $f : (a, b) \to \mathbb{R}$ is a function from an open interval $(a, b) \subset \mathbb{R}$ and $c \in (a, b)$. The idea is that we can make the values of $f(x)$ arbitrarily close to the limiting value L by making x close enough to c, though not equal to c.

As with limits of sequences we want $f(x)$ arbitrarily close to L, so we want for all $\epsilon > 0$ that we can make $|f(x) - L| < \epsilon$. In order to get this to happen we need to take x close enough to c, which requires quantification: we need a distance $\delta > 0$, depending on f and ϵ, such that making $|x - c| < \delta$ but $x \neq c$ will ensure $|f(x) - L| < \epsilon$.

Definition 26. For a function $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$ we say $\lim_{x \to c} f(x) = L$ if

$$\forall \epsilon > 0 \, \exists \, \delta > 0 \text{ such that } 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

Remark 27. The condition $0 < |x - c|$ is what ensures $x \neq c$. Our choice of $c \in (a, b)$ ensures that if $\delta < \min\{|a - c|, |b - c|\}$ then $0 < |x - c| < \delta$ implies $x \in (a, b)$ and therefore $f(x)$ makes sense.

Following the same notation as was used for limits of sequences we make the following further deifnitions. As was the case for sequences, the question of whether a limit exists can have different answers in $\mathbb{Q}$ and $\mathbb{R}$, for the same reason.

Definition 28. The limit $\lim_{x \to c} f(x)$ exists if there is L such that $\lim_{x \to c} f(x) = L$. If there is no such L then we say the limit does not exist.

As with the definition of a limit of a sequence, do not worry unduly if you do not feel any intuition for this definition; practice using the definition will eventually help you to make sense of it. It is, however, critical that you know the definition and get comfortable with the structure of arguments using it. As with limits of sequences, the structure of a proof is often different than the intuition needed to figure it out, but once you have it you can put the proof in fairly standard form like the following: Given $\epsilon > 0$ define $\delta > 0$ depending on ϵ and f, assume $0 < |x - c| < \delta$ and then give an argument which reasons from this assumption to the conclusion that $|f(x) - L| < \epsilon$. We illustrate with some examples.

Example 29. Problem: Show $\lim_{x \to 1} 2x - 3 = -1$.
Reasoning: For $f(x) = 2x - 3$ we need to get $\epsilon > |f(x) - (-1)| > |2x - 3 + 1| = |2(x - 1)|$ by making $0 < |x - 1| < \delta$. If we put $\delta = \epsilon/2$ then $|x - 1| < \delta = \epsilon/2$ so $2|x - 1| < \epsilon$, which works.
Proof: Given $\epsilon > 0$ let $\delta = \epsilon/2$ and suppose $0 < |x - 1| < \delta$. Then $|f(x) - (-1)| = |2(x - 1)| < 2\delta = \epsilon$

In the previous problem we did not really need $0 < |x - 1|$ because at $x = c$ we had $|2x - 3 - (-1)| = 0$. However it is not so difficult to give a similar example where the $0 < |x - c|$ condition is essential.

Example 30. Problem: Show that $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2} = 1$.
Reasoning: Factoring the numerator we have $x^2 + 5x + 6 = (x + 2)(x + 3)$. For $x \neq -2$ we can

then cancel the common factor of $x+2$ from the denominator and the numerator, so $|f(x)-1| = |x+3-1| = |x+2|$, and we must prove this is less than ϵ when $0 < |x+2| < \delta$. Clearly it suffices to set $\delta = \epsilon$.
Proof: Given $\epsilon > 0$ let $\delta = \epsilon$ and suppose $0 < |x+2| < \delta$. Factoring the function in the limit we obtain

$$\Big|\frac{x^2+5x+6}{x+2} - 1\Big| = \Big|\frac{(x+2)(x+3)}{x+2} - 1\Big| = |x+3-1| = |x+2| < \delta = \epsilon$$

where the cancellation is justified by the fact that $|x+2| > 0$. This proves the result.

Example 31. Problem: Show $\lim_{x\to 2} x^2 = 4$.
Reasoning: We need to show $|x^2-4| < \epsilon$ if $|x-2|$ is small enough. The essential point is that x^2-4 contains a factor of $x-2$. Specifically, $x^2-4 = (x-2)(x+2)$, which implies $|x^2-4| = |x-2||x+2|$. If we knew $|x-2| < \delta$ then $2-\delta < x < 2+\delta$ so $|x+2| < 4+\delta$, which would make $|x^2-4| < (4+\delta)|x-2| < (4+\delta)\delta$. We'd want $(4+\delta)\delta \le \epsilon$ to complete the argument. It is a bit messy to choose δ using the quadratic formula, so instead we just require that $\delta < 1$ so $(4+\delta)\delta < 5\delta$ and choose δ so $5\delta \le \epsilon$. To get our two conditions to be true we take $\delta = \min\{1, \epsilon/5\}$.
Proof: Given $\epsilon > 0$ let $\delta = \min\{1, \epsilon/5\}$. Then $|x^2-4| = |x+2||x-2| < (4+\delta)|x-2| \le 5\delta \le \epsilon$.

The idea that if $f(x)$ is a polynomial then $|f(x)-L|$ should contain a factor of $|x-c|$ is applicable in general. Here is an example that looks a little different.

Example 32. Problem: Show $\lim_{x\to 1} x^4 - 3x^3 + x^2 - 1 = -2$.
Reasoning: We need to show $|x^4 - 3x^3 + x^2 - 1 - (-2)|$ is small when $|x-1|$ is small, and have just said that this should occur because the first expression contains a factor $|x-1|$. Let's try to find the factor by long division of polynomials. (I may not use your favorite format for such long division, but you can always do it yourself. however you prefer.)

$$\begin{aligned}
x^4 - 3x^3 + x^2 - 1 - (-2) &= x^4 - 3x^3 + x^2 + 1 \\
&= x^3(x-1) - 2x^3 + x^2 + 1 \\
&= x^3(x-1) - 2x^2(x-1) - x^2 + 1 \\
&= (x^3 - 2x^2)(x-1) - x(x-1) - x + 1 \\
&= (x^3 - 2x^2 - x)(x-1) - 1(x-1) \\
&= (x^3 - 2x^2 - x - 1)(x-1)
\end{aligned}$$

Note that although we had to do this working step-by-step to get the factorizations, in the proof we only need to state what the factorization is (we don't need to include all steps) because the reader could verify it quickly by multiplying out the factors.

Now we want to make sure the factor multiplying $(x-1)$ is not too big. As in the previous problem it helps to insist that $\delta < 1$. When $|x-1| < \delta < 1$ we have $0 < x < 2$, so $|x|^3 < 8$, $2|x|^2 < 8$ and $|x| < 2$ from which we have the following (admittedly not very good) bound using the triangle inequality:

$$|x^3 - 2x^2 - x - 1| \le |x|^3 + 2|x|^2 + |x| + 1 < 8 + 8 + 2 + 1 = 19.$$

The reason we did not try to get a better bound in this expression is that it does not matter. If we take $\delta \le 19\epsilon$ then we will still get the result we seek. To fit our two conditions on δ we use a minimum as usual.
Proof: Given $\epsilon > 0$ let $\delta = \min\{1, \epsilon/19\}$. If $0 < |x-1| < \delta$ then $x \in (0,2)$ so $|x| < 2$ and we can compute

$$|x^4 - 3x^3 + x^2 - 1 - (-2)| = |x^3 - 2x^2 - x - 1||x-1| \le \big(|x|^3 + 2|x|^2 + |x| + 1\big)|x-1| < 19\delta \le \epsilon.$$

Sometimes one has a rational function where the factors in the denominator do not cancel with terms from the numerator. In this case we can still factor out a copy of $|x-c|$ but will be left bounding a rational function rather than a polynomial. The important thing is that the bounds for the terms must say the terms in the denominator stay away from 0 and that the terms in the numerator stay away from ∞.

Example 33. Problem: Show that $\lim_{x\to 0}\dfrac{x^3-x^2}{x^2+x+3}=0$.
Reasoning: We need to show that $\left|\dfrac{x^3-x^2}{x^2+x+3}\right|<\epsilon$ when $|x|<\delta$. Clearly we should factor out one or more powers of x, but one will suffice. So if we wrote

$$\Bigl|\frac{x^3-x^2}{x^2+x+3}\Bigr|\le |x|^2\frac{|x|+1}{|x^2+x+3|}$$

we see it is enough to have an upper bound for the factor with the fraction when $|x|<\delta$. An upper bound for a fraction uses a lower bound for the denominator, so we want to find a number d, so $|x^2+x+3|>d$. We'd expect this expression to be close to 3 if $|x|$ is very small. One option to show this occurs is to use the form of the triangle inequality in Lemma 3, because with $a=x^2+x+3$ and $b=3$ we obtain (also using the usual triangle inequality and $|x|<\delta$ at the end:

$$\bigl||x^2+x+3|-3\bigr|\le |x^2+x|\le \delta^2+\delta$$

If we knew $\delta\le 1$ then this is no more than $2\delta\le 2$, which gives $|x^2+x+3|\ge 1$. Then we'd get

$$\Bigl|\frac{x^3-x^2}{x^2+x+3}\Bigr|\le |x|^2\frac{|x|+1}{|x^2+x+3|}\le 2|x|^2<2\delta^2\le 2\delta$$

where at the end we again used $\delta\le 1$. Taking $2\delta<\epsilon$ will finish the proof.
Proof: Given $\epsilon>0$ let $\delta=\min\{1,\epsilon/2\}$. If $0<|x|<\delta$ then

$$\bigl||x^2+x+3|-3\bigr|\le |x^2+x|\le \delta^2+\delta\le 2$$

so that $|x^2+x+3|\ge 1$. Then we may compute

$$\Bigl|\frac{x^3-x^2}{x^2+x+3}\Bigr|\le |x|^2\frac{|x|+1}{|x^2+x+3|}\le 2|x|^2<2\delta^2<2\delta<\epsilon$$

to complete the proof.

Proving a limit does not exist frequently proceeds in the way we did for sequences. Apply contradiction, use the hypothesis of the existence of the limit to show that the function gets within ϵ of two values that are separated by distance more than 2ϵ and obtain a contradiction by the triangle inequality. All that is different is that in the sequence case the function values were points arbitrarily far out in the sequence, whereas for a limit of a function they are values $f(x)$ that occur for x very close to c. Here are two examples.

Example 34. Problem: Show that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.
Reasoning: If $x>0$ then $|x|/x=x/x=1$. If $x<0$ then $|x|/x=-x/x=-1$, so we should be able to use the same approach as in Example 11. Suppose the limit exists and is L, take $\epsilon=1$ and use the limit definition to find δ so $0<|x|<\delta$ implies $|f(x)-L|<\epsilon$. Then for any x_1, x_2 with $|x_1|<\delta$, $|x_2|<\delta$ we must have $|f(x_1)-f(x_2)|=|f(x_1)-L-(f(x_2)-L)|<2\epsilon=2$ by the triangle inequality. But if $x_1>0$ then $f(x_1)=1$ and if $x_2<0$ then $f(x_2)=-1$, so if we can find such numbers with size less than δ we get the contradiction $2=|1-(-1)|=|f(x_1)-f(x_2)|<2$. One way to get these numbers is just to take $x_1=\delta/2$ and $x_2=-\delta/2$.
Proof: Suppose the limit exists and is equal to L. Let $f(x)=|x|/x$ when $x\neq 0$. Given $\epsilon=1$ take

$\delta > 0$ so $0 < |x| < \delta$ implies $|f(x) - L| < \epsilon = 1$. Let $x_1 = \delta/2$ and $x_2 = -\delta/2$. Observe $f(x_1) = 1$ and $f(x_2) = -1$. Both $|x_1| < \delta$ and $|x_2| < \delta$, so

$$2 = |1-(-1)| = |f(x_1) - f(x_2)| = |f(x_1) - L - (f(x_2) - L)| \le |f(x_1) - L| + |f(x_2) - L| < \epsilon + \epsilon = 2$$

which gives a contradiction.

Example 35. Problem: Show that $\lim_{x\to 0} \frac{1}{x}$ does not exist.
Reasoning: We see that $\frac{1}{x}$ is large when x is small. It seems likely that we can get it to output values that have arbitrary separation, just by taking, eg, inputs x and $x/2$. This gives $f(x/2) - f(x) = (2/x) - (1/x) = \frac{1}{7}x$. If we put $x < 1$ then this difference is larger than 1. The usual way to get the contradiction is to take ϵ so 2ϵ is the separation of the function values, so lets try $\epsilon = 1/2$.
Proof: In order to obtain a contradiction, suppose there is L so $\lim_{x\to 0} \frac{1}{x} = L$. Write $f(x) = \frac{1}{x}$. With $\epsilon = \frac{1}{2}$ take $\delta > 0$ so $0 < |x| < \delta$ implies $|f(x) - L| < \epsilon$. If $0 < x < \min\{1, \delta\}$ then also $0 < \frac{x}{2} < \delta$, so that

$$\frac{1}{x} = \left|\frac{2}{x} - \frac{1}{x}\right| = \left|f(\frac{x}{2}) - f(x)\right| \le \left|f(\frac{x}{2}) - L\right| + \left|f(x) - L\right| < 2\epsilon = 1.$$

This gives a contradiction because $0 < x < 1$.

EXERCISES FOR CHAPTER 12

Section 12.1: Limits of Sequences

12.1. Give an example of a sequence that is not expressed in terms of trigonometry but whose terms are exactly those of the sequence of $\{\cos(n\pi)\}$.

12.2. Give an example of two sequences different from the sequence $\{n^2 - n! + |n-2|\}$ whose first three terms are the same as those of $\{n^2 - n! + |n-2|\}$.

12.3. Prove that the sequence $\left\{\frac{1}{2n}\right\}$ converges to 0.

12.4. Prove that the sequence $\left\{\frac{1}{n^2+1}\right\}$ converges to 0.

12.5. Prove that the sequence $\left\{1 + \frac{1}{2^n}\right\}$ converges to 1.

12.6. Prove that the sequence $\left\{\frac{n+2}{2n+3}\right\}$ converges to $\frac{1}{2}$.

12.7. By definition, $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$, there exists a positive integer N such that if n is an integer with $n > N$, then $|a_n - L| < \epsilon$. By taking the negation of this definition, write out the meaning of $\lim_{n\to\infty} a_n \neq L$ using quantifiers. Then write out the meaning of $\{a_n\}$ diverges using quantifiers.

12.8. Show that the sequence $\left\{n^4\right\}$ diverges to infinity.

12.9. Show that the sequence $\left\{\frac{n^5+2n}{n^2}\right\}$ diverges to infinity.

12.10. (a) Prove that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 2\sqrt{n}$ for every positive integer n.
(b) Let $s_n = \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \cdots + \frac{1}{n^2}$ for each $n \in \mathbf{N}$. Prove that the sequence $\{s_n\}$ converges to 0.

12.11. Prove that if a sequence $\{s_n\}$ converges to L, then the sequence $\{s_{n^2}\}$ also converges to L.

Section 12.2: Infinite Series

12.12. Prove that the series $\sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$ converges and determine its sum by

(a) computing the first few terms of the sequence $\{s_n\}$ of partial sums and conjecturing a formula for s_n;
(b) using mathematical induction to verify that your conjecture in (a) is correct;
(c) completing the proof.

12.13. Prove that the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges and determine its sum by

(a) computing the first few terms of the sequence $\{s_n\}$ of partial sums and conjecturing a formula for s_n;
(b) using mathematical induction to verify that your conjecture in (a) is correct;
(c) completing the proof.

12.14. The terms $a_1, a_2, a_3, \cdots$ of the series $\sum_{k=1}^{\infty} a_k$ are defined recursively by $a_1 = \frac{1}{6}$ and

$$a_n = a_{n-1} - \frac{2}{n(n+1)(n+2)}$$

for $n \geq 2$. Prove that $\sum_{k=1}^{\infty} a_k$ converges and determine its value.

12.15. Prove that the series $\sum_{k=1}^{\infty} \frac{k+3}{(k+1)^2}$ diverges to infinity.

12.16. (a) Prove that if $\sum_{k=1}^{\infty} a_k$ is a convergent series, then $\lim_{n\to\infty} a_n = 0$.
(b) Show that the converse of the result in (a) is false.

12.17. Let $\sum_{k=1}^{\infty} a_k$ be an infinite series whose sequence of partial sums is $\{s_n\}$ where $s_n = \frac{3n}{4n+2}$.

(a) What is the series $\sum_{k=1}^{\infty} a_k$?
(b) Determine the sum s of $\sum_{k=1}^{\infty} a_k$ and prove that $\sum_{k=1}^{\infty} a_k = s$.

Section 12.3: Limits of Functions

12.18. Give an $\epsilon - \delta$ proof that $\lim_{x\to 2}\left(\frac{3}{2}x+1\right)=4$.

12.19. Give an $\epsilon - \delta$ proof that $\lim_{x\to -1}(3x-5)=-8$.

12.20. Give an $\epsilon - \delta$ proof that $\lim_{x\to 2}(2x^2-x-5)=1$.

12.21. Give an $\epsilon - \delta$ proof that $\lim_{x\to 2} x^3=8$.

12.22. Determine $\lim_{x\to 1}\frac{1}{5x-4}$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.23. Give an $\epsilon - \delta$ proof that $\lim_{x\to 3}\frac{3x+1}{4x+3}=\frac{2}{3}$.

12.24. Determine $\lim_{x\to 3}\frac{x^2-2x-3}{x^2-8x+15}$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.25. Show that $\lim_{x\to 0}\frac{1}{x^2}$ does not exist.

12.26. The function $f:\mathbf{R}\to\mathbf{R}$ is defined by

$$f(x)=\begin{cases}1 & x<3\\ 1.5 & x=3\\ 2 & x>3.\end{cases}$$

(a) Determine whether $\lim_{x\to 3} f(x)$ exists and verify your answer.
(b) Determine whether $\lim_{x\to \pi} f(x)$ exists and verify your answer.

12.27. A function $g:\mathbf{R}\to\mathbf{R}$ is **bounded** if there exists a positive real number B such that $|g(x)|<B$ for each $x\in\mathbf{R}$.

(a) Let $g:\mathbf{R}\to\mathbf{R}$ be a bounded function and suppose that $f:\mathbf{R}\to\mathbf{R}$ and $a\in\mathbf{R}$ such that $\lim_{x\to a} f(x)=0$. Prove that $\lim_{x\to a} f(x)g(x)=0$.
(b) Use the result in (a) to determine $\lim_{x\to 0} x^2\sin\left(\frac{1}{x^2}\right)$.

12.28. Suppose that $\lim_{x\to a} f(x)=L$, where $L>0$. Prove that $\lim_{x\to a}\sqrt{f(x)}=\sqrt{L}$.

12.29. Suppose that $f:\mathbf{R}\to\mathbf{R}$ is a function such that $\lim_{x\to 0} f(x)=L$.

(a) Let $c\in\mathbf{R}$. Prove that $\lim_{x\to c} f(x-c)=L$.
(b) Suppose that f also has the property that $f(a+b)=f(a)+f(b)$ for all $a,b\in\mathbf{R}$. Use the result in (a) to prove that $\lim_{x\to c} f(x)$ exists for all $c\in\mathbf{R}$.

12.30. Let $f:\mathbf{R}\to\mathbf{R}$ be a function.

(a) Prove that if $\lim_{x\to a} f(x)=L$, then $\lim_{x\to a}|f(x)|=|L|$.
(b) Prove or disprove: If $\lim_{x\to a}|f(x)|=|L|$, then $\lim_{x\to a} f(x)$ exists.

Section 12.4: Fundamental Properties of Limits of Functions

12.31. Use limit theorems to determine the following:

(a) $\lim_{x\to 1}(x^3-2x^2-5x+8)$
(b) $\lim_{x\to 1}(4x+7)(3x^2-2)$
(c) $\lim_{x\to 2}\frac{2x^2-1}{3x^3+1}$

12.32. Use induction to prove that for every integer $n\geq 2$ and every n functions $f_1, f_2, \cdots, f_n$ such that $\lim\limits_{x\to a} f_i(x)=L_i$ for $1\leq i\leq n$,

$$\lim_{x\to a}(f_1(x)+f_2(x)+\cdots+f_n(x))=L_1+L_2+\cdots+L_n.$$

12.33. Use Exercise 12.32 to prove that $\lim_{x\to a} p(x) = p(a)$ for every polynomial $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$.

12.34. Prove that if $f_1, f_2, \ldots, f_n$ are any $n \geq 2$ functions such that $\lim_{x\to a} f_i(x) = L_i$ for $1 \leq i \leq n$, then

$$\lim_{x\to a}(f_1(x)\cdot f_2(x)\cdots f_n(x)) = L_1 \cdot L_2 \cdots L_n.$$

Section 12.5: Continuity

12.35. The function $f : \mathbf{R} - \{0, 2\} \to \mathbf{R}$ is defined by $f(x) = \frac{x^2-4}{x^3-2x^2}$. Use limit theorems to determine whether f can be defined at 2 such that f is continuous at 2.

12.36. The function f defined by $f(x) = \frac{x^2-9}{x^2-3x}$ is not defined at 3. Is it possible to define f at 3 such that f is continuous there? Verify your answer with an $\epsilon - \delta$ proof.

12.37. Let $f : \mathbf{R} \to \mathbf{Z}$ be the ceiling function defined by $f(x) = \lceil x \rceil$. Give an $\epsilon - \delta$ proof that if a is a real number that is not an integer, then f is continuous at a.

12.38. Show that Exercise 12.33 implies that every polynomial is continuous at every real number.

12.39. Prove that the function $f : [1, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x-1}$ is continuous at $x = 10$.

12.40. (a) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

In particular, $f(0) = 0$. Prove or disprove: f is continuous at $x = 0$.

(b) The problem in (a) should suggest another problem to you. State and solve such a problem.

Section 12.6: Differentiability

12.41. The function $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2$. Determine $f'(3)$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.42. The function $f : \mathbf{R} - \{-2\} \to \mathbf{R}$ is defined by $f(x) = \frac{1}{x+2}$. Determine $f'(1)$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.43. The function $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^3$. Determine $f'(a)$ for $a \in \mathbf{R}^+$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.44. The function $f : \mathbf{R} \to \mathbf{R}$ is defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine $f'(0)$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

ADDITIONAL EXERCISES FOR CHAPTER 12

12.45. Prove that the sequence $\left\{\frac{n+1}{3n-1}\right\}$ converges to $\frac{1}{3}$.

12.46. Prove that $\lim_{n\to\infty} \frac{2n^2}{4n^2+1} = \frac{1}{2}$.

12.47. Prove that the sequence $\{1 + (-2)^n\}$ diverges.

12.48. Prove that $\lim_{n\to\infty} (\sqrt{n^2+1} - n) = 0$.